

A component GARCH model with time varying weights

Un modello GARCH a componenti con pesi tempo variabili

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Riassunto: Nel lavoro si propone un nuovo modello dinamico per la volatilità dei rendimenti finanziari. Quest'ultima viene modellata come combinazione convessa di due componenti GARCH non osservabili caratterizzate da diversi livelli di persistenza, dove i pesi della combinazione variano nel tempo in funzione di opportunamente scelte variabili di stato. La funzione di verosimiglianza può essere ottenuta in maniera standard attraverso la decomposizione dell'errore di previsione. Viene infine presentata un'applicazione ad una serie di rendimenti giornalieri sull'indice *S&P500*.

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1. Introduction

In the past two decades the empirical evidence coming from financial markets has shown that the pattern of response of market volatility to shocks is highly dependent on the magnitude of these shocks. In particular, in several papers (among the others see Lamoureux and Lastrapes, 1993), it has been observed how the persistence of the volatility process tends to decrease after extreme events such as those observed in October 1987 and September 2001.

The class of Markov-switching GARCH (MS-GARCH) models offers a valuable tool for modeling state dependence in the dynamics of the volatility process. However, their application is still limited by the severe difficulties arising in the estimation and identification of these models. Haas *et al.* (2004) present an overview of such limitations and an up to date state of the art of the research on MS-GARCH models.

In order to allow for time varying persistence in the volatility dynamics, we use a modification of the component GARCH model proposed by Ding and Granger (1996) in which the weights associated to the model components are time varying and can depend on adequately chosen state variables such as lagged values of the conditional standard deviation. As in the time invariant model by Ding and Granger (1996), the conditional variance still results from the convex combination of a long and a short term component but these are mixing at different rates at each time point. Moreover, likelihood based inference for the proposed model is readily available. The conditional log-likelihood function can be obtained in a straightforward manner by means of a standard prediction error decomposition and maximized using routine optimization algorithms.

The structure of the paper is as follows. Section 2 illustrates the proposed model and discusses problems related to its estimation. Section 3 presents an application of the proposed modeling approach to daily stock returns and concludes.

2. The model

Assume that u_t is a martingale difference sequence. The proposed model is given by the following equations

$$\begin{aligned} u_t &= h_t z_t \\ h_t^2 &= w_t h_{1t}^2 + (1 - w_t) h_{2t}^2 \end{aligned} \quad (1)$$

where z_t , for $t = 1, \dots, T$, is an *iid* $(0, 1)$ sequence of random variables and

$$\begin{aligned} h_{1t}^2 &= a_{01} + a_{11} u_{t-1}^2 + b_{11} h_{1,t-1}^2 \\ h_{2t}^2 &= a_{02} + a_{11} u_{t-1}^2 + b_{22} h_{2,t-1}^2, \end{aligned}$$

with (a_{0i}, a_{1i}, b_{1i}) being constant coefficients satisfying the constraints $a_{0i} > 0$, $a_{1i} \geq 0$ and $b_{1i} \geq 0$, $i = 1, 2$. Furthermore it is assumed that $w_t = w(I_{t-1})$, with $0 \leq w_t \leq 1$, where I_{t-1} denotes the set of information available at time $(t-1)$. The basic idea is to explain the state dependence features of the volatility process by modeling it as the weighted average of two components whose weights change as a function of observable state variables. In the remainder we will denote the model in (1) as a Weighted GARCH model, abbreviated WGARCH. In principle, for the function $w(\cdot)$ several choices are possible. A convenient solution is to adopt the logistic specification

$$w_t = \frac{1}{1 + \exp(\omega + \delta' x_{t-1})} \quad (2)$$

where ω (1×1) and δ ($k \times 1$) are unknown constant coefficients and x_{t-1} ($k \times 1$) is a vector of explanatory variables. The proposed specification is very general and can potentially accommodate for a variety of situations in which the volatility dynamics are characterized by state dependence features. One example relates to the drop in the volatility persistence which is usually observed after extreme shocks such as stock market crashes. In this case an appropriate choice could be to set $x_{t-1} = h_{t-d}$, where d is a parameter determining the *delay* needed for the state to affect the current volatility dynamics. However, by letting the weights in (2) to depend on adequately chosen state variables, many other situations such as leverage and seasonal effects could be dealt with.

Inference for the model in (1) does not suffer from the limitations typically affecting other competing models such as MS-GARCH models. In particular, deriving the prediction error decomposition form of the log-likelihood function of model in (1) is straightforward:

$$\ell(u; \theta) = \sum_{t=1}^T \log f(u_t h_t^{-1}; \eta) - \sum_{t=1}^T \log h_t \quad (3)$$

where $u = (u_1 u_2 \dots u_T)$, $f(\cdot; \eta)$ denotes the probability density function of the standardized error z_t , which may be indexed by a parameter η , and $\theta = (\gamma', \delta')'$ is the vector of parameters to be estimated. Thus γ includes the GARCH component parameters, plus ω and η if the latter exists. The log-likelihood function in (3) can be maximized resorting to standard numerical procedures.

3. An application to the daily *S&P500* series

In this section, the proposed modeling approach has been applied to a time series of log returns on the *S&P500* index from 5/1/1971 to 15/8/2005, for a total of 8739 observations. As a state variable for determining the component weights, the lagged conditional standard deviation h_{t-d} has been considered. The model parameters have been estimated by maximum likelihood under two different assumptions for the distribution of z_t : a normal ($WGARCH_g$) and a t_η distribution ($WGARCH_t$) (Table 1). The value of d has been chosen within the interval $1 \leq d \leq 5$ selecting the model characterized by the maximum average value of the maximized log-likelihood. The serial correlation structure of the returns series was modeled as an AR(2) process whose parameters were estimated simultaneously with those of the conditional variance model. In both cases, t and normal, the volatility process results from the combination of a less persistent (first) component with a nearly integrated one. The negative sign of the δ_1 coefficient implies that the weight (w_t) of the less persistent component increases when the lagged market volatility increases. In the Gaussian case, as the market moves from turbulent to low volatility periods, the model switches almost completely from the less persistent component to the more persistent one. Differently, in the t case, neither component excludes the other and the overall volatility always results from a combination of the two regimes with non-zero weights (Figure 1). Indeed, the persistence gap between the two components is smaller for the model with t errors than with Gaussian errors. On the other hand, estimating standard GARCH(1,1) models, with both Gaussian ($GARCH_g$) and t ($GARCH_t$) errors, the resulting model is very close to an IGARCH failing to capture the dependence of the volatility persistence on market conditions. Among the estimated models, the WGARCH

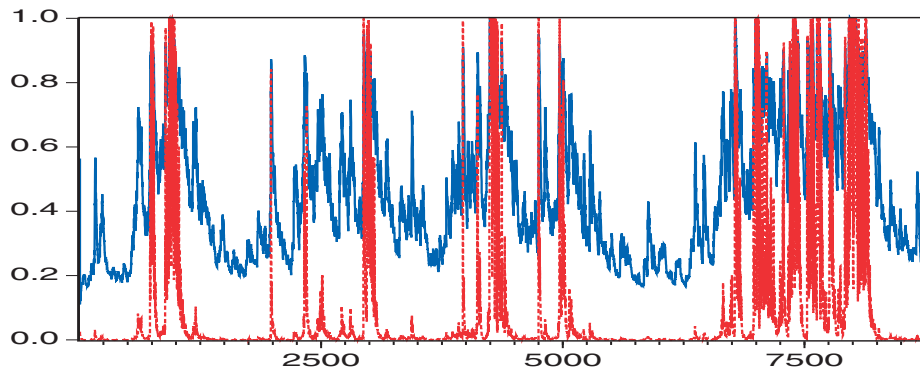


Figure 1: *Weights (w_t) for the models with Gaussian (dashed) and t (solid) errors .*

specification with t errors yields the lowest value of the Schwarz Criterion (SC) even if it fails to remove the low order autocorrelation structure of the squared residuals. The same problem affects the standard GARCH model with t errors. Finally, in order to evaluate the effectiveness of the proposed model for financial management, a VaR estimation exercise has been performed. Namely all the models considered have been applied in order to estimate the VaR for the *S&P500* series. Both short and long positions were considered at three different VaR confidence levels (0.01, 0.05, 0.10). The accuracy of VaR estimates was assessed by means of the usual Kupiec's LR test statistic (Kupiec, 1995) (Table 2).

Table 1: *ML estimates and standard errors, maximized log-likelihood (ℓ) and Schwarz Criterion (SC) for GARCH and WGARCH models.*

	$GARCH_g$	$GARCH_t$	$WGARCH_g$ ($d=1$)	$WGARCH_t$ ($d=3$)
ϕ_0	0.044 (0.009)	0.044 (0.009)	0.012 (0.008)	0.042 (0.008)
ϕ_1	0.092 (0.011)	0.082 (0.011)	0.023 (0.012)	0.081 (0.011)
ϕ_2	-0.011 (0.012)	-0.023 (0.011)	0.023 (0.011)	-0.023 (0.011)
a_{01}	0.009 (0.002)	0.006 (0.001)	0.321 (0.129)	0.143 (0.061)
a_{11}	0.067 (0.019)	0.052 (0.004)	0.380 (0.040)	0.088 (0.015)
b_{11}	0.926 (0.017)	0.942 (0.005)	0.448 (0.039)	0.851 (0.031)
a_{02}			0.004 (0.001)	0.000 (0.001)
a_{12}			0.046 (0.003)	0.006 (0.003)
b_{12}			0.951 (0.003)	0.981 (0.005)
ω			12.215 (2.136)	3.480 (0.487)
δ_1			-8.572 (1.657)	-3.499 (0.620)
η		8.043 (0.499)		8.301 (0.542)
ℓ	-11213.86	-11008.83	-11157.10	-10982.87
SC	2.5735	2.5276	2.5657	2.5269

The WGARCH model with t errors performs better than the model with Gaussian errors and, overall, the models with t errors outperform their Gaussian counterparts. However, the lack of fit in the right tail of the return distribution, at the 99th percentile, seems to suggest that the use of an asymmetric distribution could help to further improve the quality of VaR estimates.

Table 2: *VaR estimation results for long (top) and short positions (bottom): empirical coverage and LR test statistic (p -value).*

	0.10		0.05		0.01	
	$\hat{p}$	$p - value$	$\hat{p}$	$p - value$	$\hat{p}$	$p - value$
$GARCH_g$	0.094	0.049	0.046	0.126	0.013	0.005
$GARCH_t$	0.108	0.019	0.052	0.429	0.009	0.173
$WGARCH_g$	0.095	0.109	0.048	0.281	0.013	0.002
$WGARCH_t$	0.107	0.023	0.051	0.688	0.009	0.361
	0.10		0.05		0.01	
	$\hat{p}$	$p - value$	$\hat{p}$	$p - value$	$\hat{p}$	$p - value$
$GARCH_g$	0.087	0.000	0.043	0.004	0.010	0.778
$GARCH_t$	0.100	0.989	0.048	0.281	0.007	0.006
$WGARCH_g$	0.088	0.000	0.045	0.025	0.011	0.220
$WGARCH_t$	0.100	0.932	0.046	0.114	0.008	0.016

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